

# MULTIPLICITY ON A RICHARDSON VARIETY IN A COMINUSCULE $G/P$

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ABSTRACT. We show that in a cominuscule partial flag variety  $G/P$ , the multiplicity of an arbitrary point on a Richardson variety  $X_w^v = X_w \cap X^v \subset G/P$  is the product of its multiplicities on the Schubert varieties  $X_w$  and  $X^v$ .

## INTRODUCTION

Richardson varieties, named after [33], are intersections of a Schubert variety and an opposite Schubert variety inside a partial flag variety  $G/P$  ( $G$  a connected complex semi-simple group,  $P$  a parabolic subgroup). They previously appeared in [9, Ch. XIV, §4] and [36], as well as the corresponding open cells in [6]. They have since played a role in different contexts, such as equivariant K-theory [24], positivity in Grothendieck groups [3], standard monomial theory [4], Poisson geometry [8], positroid varieties [13], and their generalizations [14, 1].

On the other hand, singularities of Schubert varieties have been extensively studied in the last decades. The singular locus of Schubert varieties in Grassmannians has been determined independently in [37] and [27], and more generally in a minuscule  $G/P$  in [26]. In the full flag variety of type  $A_n$ , it has been determined independently in [2], [5], [12], and [29].

Moreover, the multiplicity of a singular point on a Schubert variety is known in several cases: when  $G/P$  is minuscule of arbitrary type, or cominuscule of type  $C_n$ , a recursive formula was given in [26]. A direct determinantal formula was given in [34] for  $G/P$  a Grassmannian; it has been subsequently interpreted in terms of non-intersecting lattice paths [17]. The multiplicity problem has also been studied in relationship with Hilbert functions and Gröbner degenerations [7, 16, 18, 23, 31, 32], as well as with  $T$ -equivariant cohomology [10, 11, 15, 20, 21, 25]. The problem of determining the multiplicity of a point in a Schubert variety in the full flag variety is more complicated; see [39, 28, 40, 41].

For Richardson varieties in a minuscule  $G/P$ , the multiplicity of a  $T$ -fixed point ( $T \subset P$  a maximal torus in  $G$ ) has been determined by Kreiman and Lakshmibai [22] (for the Gröbner point of view, see also [19] in type  $A_n$  and [38] in orthogonal types).

In this paper, we determine the multiplicity of an arbitrary point<sup>1</sup> on a Richardson variety in a cominuscule  $G/P$ .

Before stating the main result, let us fix some notation. Let  $G, P, T$  be as above, with  $G$  adjoint. Let  $X(T)$  be the character group of  $T$ ,  $R \subset X(T)$  the root system,

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<sup>1</sup>Note that unlike in the case of a Schubert variety, this no longer follows from information about  $T$ -fixed points, as pointed out in the introductions of [22] and [19].

and  $W = N_G(T)/T$  its Weyl group. Let  $B \subset G$  be a Borel subgroup such that  $T \subset B \subset P$ : it determines a system of positive roots  $R^+$  and a system of simple roots  $S$ . Denote by  $B^-$  the opposite Borel subgroup (*i.e.* such that  $B \cap B^- = T$ ).

Let  $W_P \subset W$  be the subgroup associated to  $P$  (so that  $W_G = W$  and  $W_B$  is the trivial subgroup). In the quotient  $W^P = W/W_P$ , every coset  $wP$  contains a unique minimal element for the Bruhat order  $\leq$  on  $W$ , so we shall identify  $W^P$  with the set of minimal representatives. The  $B$ -orbit (resp. the  $B^-$ -orbit) of a  $T$ -fixed point  $e_\tau = \tau P$  is called a Schubert cell (resp. an opposite Schubert cell) in  $G/P$ , and denoted by  $C_\tau$  (resp.  $C^\tau$ ). Its closure is the Schubert variety  $X_\tau$  (resp. the opposite Schubert variety  $X^\tau$ ).

If  $v, w \in W^P$ , then the intersection  $X_w^v = X_w \cap X^v$  is called a *Richardson variety*; it is non-empty if and only if  $v \leq w$  (note that Schubert varieties are the particular cases  $X_w = X_w^e$  and  $X^v = X_{w_0}^v$ , where  $e, w_0 \in W$  are the identity and the longest element, respectively).

Now assume  $P$  to be maximal, and let  $\alpha$  be the associated simple root (so that  $W_P$  is generated by the reflections  $s_\delta$  with  $\delta \in S \setminus \{\alpha\}$ ). Then  $P$  (or  $\alpha$ ) is said to be

- cominuscule if  $\alpha$  occurs with a coefficient 1 in the decomposition of the highest root of  $R^+$ ;
- minuscule if  $\alpha^\vee$  is cominuscule in the dual root system  $R^\vee$ .

The main result of this paper is the following

**Theorem 0.1.** *Assume  $P$  is cominuscule. Let  $m \in X_w^v$  be arbitrary, and denote by  $\mu_w$  (resp.  $\mu^v, \mu_w^v$ ) the multiplicity of  $m$  on  $X_w$  (resp.  $X^v, X_w^v$ ). Then*

$$(1) \quad \mu_w^v = \mu_w \mu^v.$$

This result indeed determines the multiplicities on  $X_w^v$ , since those on  $X_w$  and  $X^v$  are known: types  $A_n, D_n, E_6, E_7$  are covered by [26], Section 3 (since cominuscule is equivalent to minuscule in those types), and type  $C_n$  is covered by [26], Section 4. The only remaining case, in type  $B_n$  (*cf.* the table below), is elementary, and covered in the Appendix of the present paper for the sake of completeness.

Note that (1) is exactly the result obtained in [22] for a  $T$ -fixed point in a minuscule  $G/P$ .

To prove the theorem, we shall use a description of the multiplicity using a central projection: namely, given a projective variety  $X \subset \mathbf{P}^N$  and a point  $m \in X$ , we consider the projection  $p_m$ , of centre  $m$ , onto a hyperplane not containing  $m$ . Then the multiplicity of  $m$  on  $X$  is the difference between the degree of  $X$  and the projective degree of  $p_m$ . Note that the projective degree of  $p_m$  is zero when  $X$  is a cone. We apply this description for  $X$  the projective closure of the affine trace  $X_w^v \cap \mathcal{O}_\tau$ , where  $\mathcal{O}_\tau$  is an affine open subset of  $G/P$  identified with  $\mathbf{A}^N$ . One then needs to know whether the affine traces of  $X_w, X_v, X_w^v$  are cones or not. In this setting, we can explain why we assume that  $P$  is cominuscule:

- it implies that  $X_w \cap \mathcal{O}_\tau$  is a cone over *any* point of the cell  $C_\tau$  (though this may not be the case for  $X^v \cap \mathcal{O}_\tau$ );

- we relate the central projection  $p_m$  to a map which turns out to be a **C**-action if  $P$  is cominuscule. It is this **C**-action which allows to prove all the necessary properties for  $p_m$ .

In Section 1, we give a system of local coordinates in which  $X_w \cap \mathcal{O}_\tau$  is a cone over both  $e_\tau$  and  $m$ , and  $X^v \cap \mathcal{O}_\tau$  over  $e_\tau$ . In Section 2, we prove Theorem 0.1 assuming certain formulas for the degrees involved and that  $X^v \cap \mathcal{O}_\tau$  is not a cone over  $m$ . These assumptions are summarized in Proposition 2.1, and proved in Sections 4 and 5. The proofs are based on a **C**-action linking the central projections of centres  $m$  and  $e_\tau$ ; this action is defined and studied in Section 3.

For the convenience of the reader, we give the minuscule and cominuscule weights in the following table:

$A_n$	
$B_n$	
$C_n$	
$D_n$	
$E_6$	
$E_7$	

minuscule
  cominuscule
  both

There are no minuscule nor cominuscule fundamental weight in type  $E_8, F_4, G_2$ .

**Assumption.** For the rest of the paper, the parabolic subgroup  $P$  is assumed to be cominuscule.

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## 1. LOCAL COORDINATES

The notations are as in the Introduction. Moreover,  $R_P$  denotes the root system associated with  $P$ :

$$R^+ \setminus R_P^+ = \{\beta \in R^+ \mid U_\beta \subset R_u(P)\},$$

where  $R_u(P)$  is the unipotent radical of  $P$ , and  $U_\beta$  is the root subgroup associated with  $\beta$ .

Let  $m \in X_w^v$ . Then  $m$  lies in a Schubert cell  $C_\tau$  for some  $\tau \in W^P$ . Let

$$U_\tau^- = \prod_{\beta \in \tau(R^+ \setminus R_P^+)} U_{-\beta}$$

and  $\mathcal{O}_\tau = U_\tau^- \cdot e_\tau$ , where  $e_\tau = \tau P$ . We identify  $U_{-\beta}$  with  $\mathbf{C}$  via an isomorphism  $\theta_{-\beta} : \mathbf{C} \rightarrow U_{-\beta}$  satisfying

$$t\theta_{-\beta}(x)t^{-1} = \theta_{-\beta}\left(\frac{1}{\beta(t)}x\right)$$

for all  $t \in T$  and all  $x \in \mathbf{C}$ . Let  $N$  be the cardinality of  $R^+ \setminus R_P^+$ . We identify  $\mathcal{O}_\tau$  with the affine space  $\mathbf{A}^N$  via the isomorphism

$$(2) \quad \begin{array}{ccc} \mathbf{A}^N & \longrightarrow & \mathcal{O}_\tau \\ (x_{-\beta})_{\beta \in \tau(R^+ \setminus R_P^+)} & \mapsto & \prod_{\beta \in \tau(R^+ \setminus R_P^+)} \theta_{-\beta}(x_{-\beta}) \cdot e_\tau. \end{array}$$

(In particular,  $N$  is the dimension of  $G/P$ .)

**Lemma 1.1.** *Let  $\beta \in R$ , and  $\tau \in W^P$ . Then  $U_\beta$  fixes  $e_\tau$  if and only if  $-\beta \notin \tau(R^+ \setminus R_P^+)$ .*

*Proof.* Let  $\beta \in R$ , and  $\tau \in W^P$ . Then

$$\begin{aligned} U_\beta \cdot e_\tau = e_\tau &\iff \tau^{-1}U_\beta \tau P = P \\ &\iff U_{\tau^{-1}\beta} \subset P \\ &\iff \tau^{-1}\beta \in R^+ \text{ or } -\tau^{-1}\beta \in R_P^+ \\ &\iff -\beta \notin \tau(R^+) \text{ or } -\beta \in \tau(R_P^+) \\ &\iff -\beta \notin \tau(R^+ \setminus R_P^+). \quad \square \end{aligned}$$

**Lemma 1.2.** *The Schubert cell  $C_\tau$  is the affine subspace of  $\mathcal{O}_\tau$  defined by the vanishing of the coordinates  $x_{-\beta}$  with  $\beta \in R^+$ .*

*Proof.* Since  $B$  is the semi-direct product of  $T$  and the unipotent subgroup  $U$ , we have  $C_\tau = U \cdot e_\tau$ . Moreover, for any ordering of positive roots  $\{\beta_1, \dots, \beta_p\}$ ,

$$U = \prod_{i=1}^p U_{\beta_i}.$$

We choose an ordering such that the positive roots  $\beta$  with  $-\beta \notin \tau(R^+ \setminus R_P^+)$  appear at the end. Then, by the preceding lemma, we have:

$$C_\tau = \prod_{\substack{\beta \in \tau(R^+ \setminus R_P^+) \\ \beta < 0}} U_{-\beta} \cdot e_\tau \subset \mathcal{O}_\tau. \quad \square$$

The following lemma will be useful for the next section.

**Lemma 1.3.** *For all  $\beta, \gamma \in \tau(R^+ \setminus R_P^+)$  and for all  $x, y \in \mathbf{C}$ , the elements  $\theta_\beta(x)$  and  $\theta_\gamma(y)$  commute.*

*Proof.* We use the following expansion for the commutator (cf. [35], proposition 8.2.3):

$$\theta_\beta(x)\theta_\gamma(y)\theta_\beta(x)^{-1}\theta_\gamma(y)^{-1} = \prod_{\substack{i\beta+j\gamma \in R \\ i,j > 0}} \theta_{i\beta+j\gamma}(c_{\beta,\gamma,i,j} x^i y^j),$$

where  $c_{\beta,\gamma,i,j}$  are some constants in  $\mathbf{C}$ . Since the commutator must lie in  $U_\tau^-$ , it suffices to prove that the roots of the form  $i\beta+j\gamma$  do not lie in  $\tau(R^+ \setminus R_P^+)$ . Now,  $P$  is the parabolic subgroup associated with the simple root  $\alpha$ . Since  $\alpha$  is cominuscule, a positive root  $\delta$  lies in  $R^+ \setminus R_P^+$  if and only if  $\alpha$  occurs with coefficient 1 in the expression of  $\delta$ . Clearly,  $\alpha$  occurs with a coefficient  $i+j$  in  $\tau^{-1}(i\beta+j\gamma)$ .  $\square$

**Remark 1.4.** Identifying  $\mathcal{O}_\tau$  with  $U_\tau^-$ , it follows from Lemma 1.3 that the isomorphism of algebraic varieties (2)  $\mathbf{A}^N \rightarrow \mathcal{O}_\tau$  is also an isomorphism of unipotent groups.

**Example 1.5.** Let  $G = SL_n(\mathbf{C})$ . It is a group of type  $A_{n-1}$ . The torus  $T$  is the group of diagonal matrices of determinant 1, and the Borel subgroup  $B$  is the group of upper triangular matrices of determinant 1. The roots are denoted  $\alpha_{i,j}$ , where

$$\alpha_{i,j} : T \rightarrow \mathbf{C}^* : \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \mapsto \frac{t_i}{t_j}.$$

The positive roots are the  $\alpha_{i,j}$  with  $i < j$ , and the simple roots are the  $\alpha_i = \alpha_{i,i+1}$  ( $i = 1, \dots, n-1$ ). Let  $\omega = \omega_d$  be the fundamental weight associated with the simple root  $\alpha_d$ . The corresponding parabolic subgroup  $P$  is

$$P = \left\{ \left( \begin{array}{c|c} * & * \\ \hline 0_{(n-d) \times d} & * \end{array} \right) \right\}.$$

The group  $G$  acts transitively on the Grassmannian  $G_{d,n}$  of  $d$ -spaces in  $\mathbf{C}^n$ , and  $P$  is the isotropy subgroup of the vector space generated by  $e_1, \dots, e_d$ , where  $(e_1, \dots, e_n)$  is the canonical basis of  $\mathbf{C}^n$ . The Weyl group  $W$  of this root system is  $S_n$ , and  $W_P$  is isomorphic to  $S_d \times S_{n-d}$ , so

$$W^P = I_{d,n} = \{\mathbf{i} = i_1 \dots i_d \mid 1 \leq i_1 < i_2 < \dots < i_d \leq n\}.$$

The Lie algebra  $\mathfrak{g}$  of  $G$  is the space of traceless matrices. Let  $\mathfrak{t}$  be the Lie algebra of the torus  $T$ . We have the weight decomposition of  $\mathfrak{g}$ :

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{i \neq j} \mathbf{C} E_{i,j}$$

where  $E_{i,j}$  is the elementary matrix with a 1 on the row  $i$  and column  $j$ , and zero elsewhere. Thus, the root subgroups are given by

$$U_{\alpha_{i,j}} = \{I_n + x E_{i,j} \mid x \in \mathbf{C}\}$$

and the isomorphism  $\theta_{\alpha_{i,j}}$  is just  $x \mapsto \exp(x E_{i,j})$ . Moreover,

$$R^+ \setminus R_P^+ = \{\alpha_{i,j} \mid i \leq d < j\},$$

so in this case, Lemma 1.3 becomes an elementary matrix computation.

Returning to the general case, we denote by  $(m_{-\beta} | \beta \in \tau(R^+ \setminus R_P^+))$  the coordinates of  $m$ , that is,

$$m = \prod_{\beta \in \tau(R^+ \setminus R_P^+)} \theta_{-\beta}(m_{-\beta}).e_\tau.$$

**Notations 1.6.** We set:

$$Y_w = X_w \cap \mathcal{O}_\tau, \quad Y^v = X^v \cap \mathcal{O}_\tau, \quad Y_w^v = X_w^v \cap \mathcal{O}_\tau.$$

These sets are affine varieties, *i.e.* Zariski-closed in  $\mathcal{O}_\tau = \mathbf{A}^N$ .

We now investigate if these affine varieties are cones over  $m$ .

**Proposition 1.7.** *The varieties  $Y_w$ ,  $Y^v$  and  $Y_w^v$  are cones over  $e_\tau$ .*

*Proof.* Let  $\omega^\vee : \mathbf{C}^* \rightarrow T$  be the fundamental coweight associated to  $P$ . Since  $\omega^\vee$  is minuscule, the pairing  $\langle \omega^\vee, \gamma \rangle$  is equal to 1 if  $\gamma \in R^+ \setminus R_P^+$  (and to 0 if  $\gamma \in R_P^+$ ). Now multiplication in  $\mathbf{A}^N$  by a scalar  $\xi$  is then given by conjugation in  $U_\tau^-$  by  $\tau(\omega^\vee)(\xi)^{-1} \in T$ : indeed, for  $\beta = \tau(\gamma)$  with  $\gamma \in R^+ \setminus R_P^+$ , and for  $z \in \mathbf{C}$ , we have

$$(3) \quad \tau(\omega^\vee)(\xi)^{-1} \theta_{-\beta}(z) \tau(\omega^\vee)(\xi) = \theta_{-\beta}(\xi^{\langle \tau(\omega^\vee), \beta \rangle} z) = \theta_{-\beta}(\xi^{\langle \omega^\vee, \gamma \rangle} z) = \theta_{-\beta}(\xi z).$$

Let  $x \in Y_w$  (resp.  $x \in Y^v$ ), and  $(x_{-\beta})$  be its coordinates. Then the point that has coordinates  $(\xi x_{-\beta})$  is  $t.x$ , where  $t = \tau(\omega^\vee)(\xi) \in T$ . Therefore, this point lies in  $X_w \cap \mathcal{O}_\tau$  (resp. in  $X^v \cap \mathcal{O}_\tau$ ), since  $X_w$  (resp.  $X^v$ ) is  $T$ -stable. It follows that  $Y_w$ ,  $Y^v$ , and therefore  $Y_w^v$  are cones over  $e_\tau$ .  $\square$

**Proposition 1.8.** *The variety  $Y_w$  is a cone over  $m$ .*

*Proof.* Consider the translation that maps  $e_\tau$  to  $m$ . It is given in coordinates by  $(x_{-\beta}) \mapsto (x_{-\beta} + m_{-\beta})$ . But if  $x$  has coordinates  $(x_{-\beta})$ , then, by Remark 1.4 the point of coordinates  $(x_{-\beta} + m_{-\beta})$  corresponds to  $b.x$ , where  $b = \prod_{\beta} \theta_{-\beta}(m_{-\beta})$ . Since  $m_{-\beta} = 0$  for all  $\beta > 0$ , we have  $b \in B$  according to Lemma 1.2. Now  $b$  leaves  $Y_w$  invariant and maps  $e_\tau$  to  $m$ .  $\square$

However, the opposite Schubert variety  $Y^v$  need not be a cone over  $m$ .

**Example 1.9.** We take the same notations as in Example 1.5. In particular, using the identification  $W^P = I_{d,n}$ , we denote a Schubert variety in  $G_{d,n}$  by  $X_{i_1 \dots i_d}$ , and similarly for opposite Schubert and Richardson varieties. In the Grassmannian  $G_{3,7}$ , consider the Richardson variety  $X_{356}^{125}$ . The coordinates on the open set  $\mathcal{O}_{256}$  are parametrized by the set  $\{12, 15, 16, 32, 35, 36, 42, 45, 46, 72, 75, 76\}$  where  $ij$  stands for the root  $\alpha_{i,j}$ . More precisely, we have:

$$\begin{aligned} \mathbf{A}^{12} &\longrightarrow \mathcal{O}_{256} \\ (x_{12}, x_{15}, \dots, x_{76}) &\mapsto \begin{bmatrix} x_{12} & x_{15} & x_{16} \\ 1 & 0 & 0 \\ x_{32} & x_{35} & x_{36} \\ x_{42} & x_{45} & x_{46} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_{72} & x_{75} & x_{76} \end{bmatrix}. \end{aligned}$$

Here, a matrix between brackets actually stands for the 3-space in  $\mathbf{C}^7$  generated by its columns. The equations of  $X_{356}$  are:

$$\begin{cases} x_{72} = x_{75} = x_{76} = 0 \\ x_{42} = 0 \end{cases}$$

The equations of  $X^{125}$  are:

$$\begin{cases} x_{15}x_{36} - x_{35}x_{16} = 0 \\ x_{15}x_{46} - x_{45}x_{16} = 0 \\ x_{35}x_{46} - x_{45}x_{36} = 0 \end{cases}$$

Let

$$m = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in X_{356}^{125}.$$

We set:

$$\begin{cases} y_{16} = x_{16} - 1 \\ y_{36} = x_{36} + 1 \\ y_{ij} = x_{ij} \quad \text{if } ij \notin \{16, 36\} \end{cases}$$

The equations in these new coordinates are:

$$\begin{cases} y_{72} = y_{75} = y_{76} = 0 \\ y_{42} = 0 \end{cases}$$

for  $X_{356}$  and

$$\begin{cases} y_{15}(y_{36} - 1) - y_{35}(y_{16} + 1) = 0 \\ y_{15}y_{46} - y_{45}(y_{16} + 1) = 0 \\ y_{35}y_{46} - y_{45}(y_{36} - 1) = 0 \end{cases}$$

for  $X^{125}$ . While the equations for  $X_{356}$  remain homogeneous, those for  $X^{125}$  do not.

If  $Y^v$  is indeed a cone over  $m$ , then we have the following result. The proof is taken from [22], Remark 7.6.6.

**Proposition 1.10.** *Assume  $Y^v$  is a cone over  $m$ . Let  $\mu_w$  (resp.  $\mu^v$ ,  $\mu_w^v$ ) be the multiplicity of  $m$  on  $X_w$  (resp.  $X^v$ ,  $X_w^v$ ). Then*

$$(4) \quad \mu_w^v = \mu_w \mu^v.$$

*Proof.* In this case,  $Y_w^v = Y_w \cap Y^v$  is a cone (over  $m$ ) as well, so we may consider the projective varieties  $\mathbf{P}(Y_w)$ ,  $\mathbf{P}(Y^v)$  and  $\mathbf{P}(Y_w^v)$ , consisting of lines through  $m$ . Then  $\mu_w$  (resp.  $\mu^v$ ,  $\mu_w^v$ ) is just the degree of  $\mathbf{P}(Y_w)$  (resp.  $\mathbf{P}(Y^v)$ ,  $\mathbf{P}(Y_w^v)$ ). We conclude with Bézout's theorem since  $\mathbf{P}(Y_w)$  and  $\mathbf{P}(Y^v)$  intersect transversely (cf. [33], Corollary 1.5).  $\square$

**Assumption 1.11.** For the rest of the paper, we assume that  $Y^v$  is not a cone over  $m$ .

It is not clear however whether  $Y_w^v$  is a cone or not. This problem will be solved in Section 4.

## 2. CENTRAL PROJECTION AND PROOF OF THEOREM 0.1

We shall compute the multiplicity of a point  $m \in Y_w^v$  by relating it to degrees of projections, which requires us to work in a projective setting. More precisely, embed  $\mathbf{A}^N$  into  $\mathbf{P}^N$  via

$$\begin{aligned} \iota : \quad \mathbf{A}^N &\hookrightarrow \mathbf{P}^N = \{[\xi : x_{-\beta}]\} \\ (x_{-\beta}) &\mapsto [1 : x_{-\beta}] \end{aligned}$$

and consider the projective closures

$$Z_w = \overline{\iota(Y_w)}, \quad Z^v = \overline{\iota(Y^v)}, \quad Z_w^v = \overline{\iota(Y_w^v)}.$$

We also identify  $\mathbf{P}^{N-1}$  with the hyperplane at infinity  $\xi = 0$  and consider the central projection  $p_m : \mathbf{P}^N \rightarrow \mathbf{P}^{N-1}$ , sending any point  $x \neq m$  to the intersection of the line  $(mx)$  with  $\mathbf{P}^{N-1}$ . If  $X \subset \mathbf{P}^N$  is any projective variety and  $m \in X$ , then we have the following formula (cf. [30], Theorem 5.11):

$$(5) \quad \deg X - \text{mult}_m X = \begin{cases} \deg(p_m)|_X \deg(p_m X) & \text{if } X \text{ is not a cone over } m, \\ 0 & \text{if } X \text{ is a cone over } m, \end{cases}$$

where  $\deg X$  is the degree of  $X$ ,  $\deg(p_m)|_X$  is the degree of the rational map  $p_m$  restricted to  $X$ , and  $p_m X$  denotes the Zariski closure of  $p_m(X \setminus \{m\})$ .

### Proposition 2.1.

- (a)  $\deg Z_w^v = \deg Z_w \deg Z^v$ .
- (b)  $Z_w^v$  is not a cone over  $m$ .
- (c)  $\deg(p_m)|_{Z_w^v} = \deg(p_m)|_{Z^v}$ .
- (d)  $\deg(p_m Z_w^v) = \deg Z_w \deg(p_m Z^v)$ .

We defer the proof to Section 4.

*Proof of Theorem 0.1.* Using (5) and Proposition 2.1, we obtain

$$\begin{aligned} \mu_w^v &= \deg Z_w^v - \deg(p_m)|_{Z_w^v} \deg(p_m Z_w^v) \\ &= \deg Z_w \deg Z^v - \deg(p_m)|_{Z^v} \deg Z_w \deg(p_m Z^v) \\ &= \deg Z_w [\deg Z^v - \deg(p_m)|_{Z^v} \deg(p_m Z^v)] \\ &= \mu_w \mu^v. \quad \square \end{aligned}$$

**Remark 2.2.** In particular, this result enables us to find the singular locus of  $X_w^v$  in terms of those of  $X_w$  and  $X^v$ : the point  $m$  is smooth on  $X_w^v$  if and only if  $\mu_w^v = 1$  if and only if  $\mu_w = \mu^v = 1$ , that is, if and only if  $m$  is smooth on both  $X_w$  and  $X^v$ . Note that this may also be seen more directly, using the fact that  $X_w$  and  $X^v$  intersect properly and transversely at any point at which  $\mu_w = \mu^v = 1$  (cf. [33] Corollary 1.5, or [1] Corollary 2.9).



3.  $\mathbf{C}$ -ACTION ON  $G/P$ 

In this section, we introduce the main tool that will permit us to prove Proposition 2.1 in the next section. Let  $e_\tau, m \in \mathcal{O}_\tau$  be as before: we shall construct an action of (the additive group)  $\mathbf{C}$  on  $G/P$  for which  $e_\tau$  and  $m$  are in the same orbit.

Consider first the map

$$\begin{aligned} \varphi^* : \mathbf{C}^* &\rightarrow B \\ \xi &\mapsto \varphi_\xi = \tau(\omega^\vee)(\xi)^{-1} b \tau(\omega^\vee)(\xi), \end{aligned}$$

where  $b \in B \cap \mathcal{U}_\tau^-$  is the element defined in the proof of Proposition 1.8. The computation (3) shows that this map extends to a group homomorphism  $\varphi : \mathbf{C} \rightarrow B$ . The natural  $B$ -action on  $G/P$  thus induces a  $\mathbf{C}$ -action:

$$\begin{aligned} \Phi : \mathbf{C} \times G/P &\rightarrow G/P \\ (\xi, x) &\mapsto \varphi_{-\xi}.x \end{aligned}$$

Moreover,  $\mathcal{O}_\tau$  is invariant under this action (again by (3)). Actually,  $\mathbf{C}$  acts on  $\mathcal{O}_\tau = \mathbf{A}^N$  by translations: indeed, we get the following commutative diagram

$$(6) \quad \begin{array}{ccc} \mathbf{C} \times \mathbf{A}^N & \xrightarrow{\Phi} & \mathbf{A}^N \\ \downarrow f & & \downarrow p_{e_\tau} \\ \mathbf{P}^N & \xrightarrow[p_m]{} & \mathbf{P}^{N-1} \end{array} \quad \begin{array}{ccc} (\xi, x_{-\beta}) & \xrightarrow{\Phi} & (x_{-\beta} - \xi m_{-\beta}) \\ \downarrow f & & \downarrow p_{e_\tau} \\ [\xi : x_{-\beta}] & \xrightarrow[p_m]{} & [0 : x_{-\beta} - \xi m_{-\beta}] \end{array}$$

Let us now restrict to  $Y_w^v$ : since it is a cone over  $e_\tau$ , a point  $[\xi : x]$  lies in  $Z_w^v$  if and only if  $x \in Y_w^v$ . It follows that  $f(\mathbf{C} \times Y_w^v) = Z_w^v$ . Thus, the commutative diagram (6) restricts to

$$(7) \quad \begin{array}{ccc} \mathbf{C} \times Y_w^v \setminus \{(\xi, \xi m_{-\beta}) \mid \xi \in \mathbf{C}\} & \xrightarrow{\Phi} & \Phi(\mathbf{C} \times Y_w^v) \setminus \{e_\tau\} \\ \downarrow f & & \downarrow p_{e_\tau} \\ Z_w^v \setminus \{m\} & \xrightarrow[p_m]{} & \mathbf{P}^{N-1} \end{array}$$

**Remark 3.1.** Since (6) is a fibre product diagram, any fibre  $\Phi^{-1}(\lambda y)$  (for  $\lambda \neq 0$  and  $[y] \in \mathbf{P}^{N-1}$ ) is mapped isomorphically via  $f$  to the fibre  $p_m^{-1}([y])$ . Since we have the equalities  $f(\mathbf{C} \times Y_w) = Z_w$ ,  $f(\mathbf{C} \times Y^v) = Z^v$ ,  $f(\mathbf{C} \times Y_w^v) = Z_w^v$  and  $\mathbf{C} \times Y_w = f^{-1}(Z_w)$ ,  $\mathbf{C} \times Y^v = f^{-1}(Z^v)$ ,  $\mathbf{C} \times Y_w^v = f^{-1}(Z_w^v)$ , the fibres of  $\Phi|_{\mathbf{C} \times Y_w}$ ,  $\Phi|_{\mathbf{C} \times Y^v}$ ,  $\Phi|_{\mathbf{C} \times Y_w^v}$  over a point  $\lambda y$  are isomorphic to the fibres of  $p_m|_{Z_w}$ ,  $p_m|_{Z^v}$ ,  $p_m|_{Z_w^v}$  over the point  $[y]$ .

In the next section, this remark will allow us to relate the degree of  $p_m$  in diagram (7) to that of  $\Phi$ .

## 4. PROOF OF PROPOSITION 2.1

**Proof of (a).** Since  $Y_w$ ,  $Y^v$ , and  $Y_w^v$  are (affine) cones over  $e_\tau$ , it is clear that  $Z_w^v = Z_w \cap Z^v$ . In addition, this intersection is proper and generically transverse ([33], Corollary 1.5), hence  $\deg Z_w^v = \deg Z_w \deg Z^v$  by Bézout's theorem.

**Notations 4.1.** We denote by  $F_w^v$  the closure in  $\mathbf{A}^N$  of  $\Phi(\mathbf{C} \times Y_w^v)$ , and by  $d_w^v$  the degree of  $p_m : Z_w^v \setminus \{m\} \rightarrow p_m Z_w^v$  whenever it makes sense (*i.e.* when  $Z_w^v$  is not a cone). We define  $F_w, F^v, d^v$  in a similar way.

**Proposition 4.2.** *When defined, the degree  $d_w^v$  is equal to the degree of  $\Phi : \mathbf{C} \times Y_w^v \rightarrow F_w^v$ .*

*Proof.* This follows from Remark 3.1.  $\square$

**Lemma 4.3.** *The following properties are equivalent:*

- $Z_w^v$  is a cone over  $m$ ,
- $F_w^v = Y_w^v$ ,
- every fibre of  $\Phi : \mathbf{C} \times Y_w^v \rightarrow F_w^v$  has dimension 1.

*In particular, they are true for  $v = e$ , hence  $F_w = Y_w = \Phi(\mathbf{C} \times Y_w)$ .*

*Proof.* By Remark 3.1, we see that the dimension of a generic fibre of  $\Phi$  equals the dimension of a generic fibre of  $p_m$ . Now  $Z_w^v$  is a cone over  $m$  if and only if every fibre of  $p_m$  has dimension 1, if and only if  $\dim F_w^v = \dim Y_w^v$ . But  $Y_w^v = \Phi(0 \times Y_w^v) \subset F_w^v$  and the varieties  $Y_w^v$  and  $F_w^v$  are irreducible, so  $Z_w^v$  is a cone over  $m$  if and only if  $F_w^v = Y_w^v$ .  $\square$

**Proof of (b) and (c).** By Proposition 4.2, it suffices to compare the degree  $d^v$  of  $\Phi^v : \mathbf{C} \times Y^v \rightarrow F^v$  with the degree  $d_w^v$  of  $\Phi_w^v : \mathbf{C} \times Y_w^v \rightarrow F_w^v$ . First, the fibre of a point  $x \in G/P$  for  $\Phi$  is

$$\Phi^{-1}(x) = \{(\xi, \Phi(-\xi, x)) \mid \xi \in \mathbf{C}\}.$$

In particular, a point lies in  $\text{Im}(\Phi^v)$  (resp. in  $\text{Im}(\Phi_w^v)$ ) if and only if its  $\mathbf{C}$ -orbit meets  $Y^v$  (resp.  $Y_w^v$ ). There exists an open set  $\Omega^v$  of  $F^v$  in which the fibre of every point  $y$  consists of  $d^v$  points. Then  $d^v$  is just the number of points in the  $\mathbf{C}$ -orbit of  $y$  that belong to  $Y^v$ . Now set  $y = (y_{-\beta})_{\beta \in \tau(R^+ \setminus R_P^+)}$  and let

$$c = \prod_{\substack{\beta \in \tau(R^+ \setminus R_P^+) \\ \beta < 0}} \theta_{-\beta}(y_{-\beta}) \quad c^- = \prod_{\substack{\beta \in \tau(R^+ \setminus R_P^+) \\ \beta > 0}} \theta_{-\beta}(-y_{-\beta}),$$

so we have  $c.e_\tau = c^-.y =: x$ . Since  $c \in B$ ,  $x \in C_\tau \subset Y_w$ . Now  $c^-$  commutes with  $\varphi_\xi$  for all  $\xi \in \mathbf{C}$ , hence every point in  $c^-(\Omega^v)$  has a  $\mathbf{C}$ -orbit which meets  $Y^v$  in exactly  $d^v$  points. In particular,  $F_w^v \neq Y_w^v$ , since otherwise every fibre of  $\Phi_w^v$  would have dimension 1 (by Lemma 4.3), which is not the case for the fibre of  $x$ . This already shows (b), so it makes sense to talk about the degree  $d_w^v$  of  $\Phi_w^v$ . Thus, let  $\Omega_w^v$  be an open set of  $F_w^v$  such that for every point  $z$  in  $\Omega_w^v$ , the fibre of  $z$  consists of  $d_w^v$  points. Since  $x \in c^-(\Omega^v)$ ,  $c^-(\Omega^v) \cap F_w^v$  and  $\Omega_w^v$  are non-empty open sets of the irreducible variety  $F_w^v$ , so they must meet. Taking  $z$  in this intersection, we see that  $d_w^v = d^v$ , which shows (c).  $\square$

**Proposition 4.4.** *The intersection  $F_w \cap F^v$  is proper and transverse on an open set of  $F_w^v$ .*

*Proof.* The transversality of the intersection  $F_w \cap F^v$  on a generic point in  $F_w^v$  follows from the transversality of the intersection of a direct Schubert variety and an opposite Schubert variety. More precisely, let  $(F_w)_{sm}$  be the open set of smooth points of  $F_w$ . Taking a point smooth on  $Y_w^v$  shows that  $\Omega_w = (F_w)_{sm} \cap F_w^v$  is a non-empty open set of  $F_w^v$ . Let  $(F^v)_{sm}$  be the open set of smooth points of  $F^v$ . Again,  $\Omega^v = (F^v)_{sm} \cap F_w^v \neq \emptyset$ . Indeed, take a smooth point  $x$  of  $F^v$  belonging to  $\Phi(\mathbf{C} \times Y^v)$ . We have seen in the previous proof that from  $x$  we can construct

an isomorphism  $c^-$  of  $F^v$  mapping  $x$  to a point of  $F_w^v$ , which thus remains smooth on  $F^v$ . The two non-empty open subsets  $\Omega_w$  and  $\Omega^v$  of the irreducible variety  $F_w^v$  have a non-empty intersection  $\Omega_w^v$ . Now  $O_w^v = \Phi^{-1}(\Omega_w^v) \cap (\mathbf{P}^1 \times Y_w^v)_{sm} \neq \emptyset$  since  $\mathbf{P}^1 \times Y_w^v$  is irreducible. We claim that  $\Phi : O_w^v \rightarrow \Omega_w^v$  is dominant. Indeed, we must show that every open subset  $U$  of  $\Omega_w^v$  meets  $\Phi(O_w^v)$ . Since  $U$  is open in  $F_w^v$ ,  $U \cap \Phi(\mathbf{C} \times Y_w^v) \neq \emptyset$ . So it makes sense to talk about  $\Phi^{-1}(U)$ , which is an open set of  $\mathbf{C} \times Y_w^v$ . Thus,  $\Phi^{-1}(U) \cap O_w^v \neq \emptyset$ , which implies  $U \cap \Phi(O_w^v) \neq \emptyset$ . Since  $\Phi : O_w^v \rightarrow \Omega_w^v$  is dominant, we know that  $\Phi(O_w^v)$  contains a non-empty open set  $\Omega$  of  $\Omega_w^v$ . Let us summarize the properties of  $\Omega$ : it is a non-empty open subset of  $F_w^v$ , whose every point  $y$  is smooth in both  $F_w$  and  $F^v$ , and  $y = \Phi(p)$  with  $p$  smooth in  $\mathbf{C} \times Y_w^v$ , so  $p$  is smooth in both  $\mathbf{C} \times Y_w$  and  $\mathbf{C} \times Y^v$ .

Let  $y = \Phi(p) \in \Omega$  be such a point. We view the map  $\Phi : \mathbf{C} \times \mathbf{A}^N \rightarrow \mathbf{A}^N : (\xi, x) \mapsto \varphi_{-\xi}.x$  as a map  $\Phi : \mathbf{C}^{N+1} \rightarrow \mathbf{C}^N$ . It is linear and surjective. Thus,

$$\begin{aligned} \mathbf{C}^N &\supset T_y(F_w) + T_y(F^v) \supset d\Phi_p(T_p(\mathbf{C} \times Y_w)) + d\Phi_p(T_p(\mathbf{C} \times Y^v)) \\ &\supset d\Phi_p(\mathbf{C} \oplus (T_p Y_w + T_p Y^v)) \\ &\supset d\Phi_p(\mathbf{C} \oplus \mathbf{C}^N) \\ &\supset \mathbf{C}^N. \end{aligned}$$

This transversality result proves that the intersection is proper: indeed, on one hand,  $\dim(F_w \cap F^v) \geq \dim(F_w) + \dim(F^v) - N$ , but on the other hand,

$$\begin{aligned} \dim(F_w \cap F^v) &\leq \dim(T_y(F_w \cap F^v)) \leq \dim(T_y F_w \cap T_y F^v) \\ &\leq \dim(T_y F_w) + \dim(T_y F^v) - \dim(T_y F_w + T_y F^v) \\ &\leq \dim(F_w) + \dim(F^v) - N. \quad \square \end{aligned}$$

**Proposition 4.5.** *We have the equality  $F_w^v = F_w \cap F^v$ . In particular, the intersection  $F_w \cap F^v$  is generically transverse.*

This result will be proved in the next section.

**Proof of (d).** Since  $y = \Phi(\xi, x)$  implies  $zy = \Phi(z\xi, zx)$  for all  $z \in \mathbf{C}$ ,  $\Phi(\mathbf{C} \times Y_w^v)$  is a cone over  $e_\tau$ , and so is its closure  $F_w^v$ . But by the commutative diagram (7),

$$p_{e_\tau}(F_w^v \setminus \{e_\tau\}) \subset \overline{p_{e_\tau}(\Phi(\mathbf{C} \times Y_w^v) \setminus \{e_\tau\})} = p_m Z_w^v.$$

Comparing dimensions, we see that  $p_{e_\tau} F_w^v = p_m Z_w^v$ , i.e.  $p_m Z_w^v$  is the projective variety at infinity of the cone  $F_w^v$ . In particular,  $\deg(p_m Z_w^v) = \deg(F_w^v)$ , and similarly  $\deg(p_m Z_w) = \deg(F_w)$  and  $\deg(p_m Z^v) = \deg(F^v)$ . Equality (d) now follows from Proposition 4.4 and Bézout's theorem, noting that  $\deg(p_m Z_w) = \deg(Z_w)$ .  $\square$

## 5. PROOF OF PROPOSITION 4.5

Since  $\Phi(\mathbf{C} \times Y_w^v) \subset \Phi(\mathbf{C} \times Y_w) \cap \Phi(\mathbf{C} \times Y^v)$ , we obtain  $F_w^v \subset F_w \cap F^v$ . Moreover, the first inclusion is an equality: indeed, if  $z = \Phi(\xi, x) \in Y_w$  with  $\xi \in \mathbf{C}$ ,  $x \in Y^v$ , then  $x = \Phi(-\xi, z) \in Y_w$  since  $\Phi(\mathbf{C} \times Y_w) = Y_w$ , so  $z = \Phi(\xi, x) \in \Phi(\mathbf{C} \times Y_w^v)$ .

However, the inclusion  $F_w \cap F^v \subset F_w^v$  requires some work. Let  $\mathcal{U} = \{(\xi, x, \Phi(\xi, x)) \mid \xi \in \mathbf{C}, x \in G/P\}$  and  $\Gamma$  be its closure in  $\mathbf{P}^1 \times G/P \times G/P$  (so  $\Gamma$  is the graph of  $\Phi$  viewed

as a rational map). We have a commutative diagram:

$$\begin{array}{ccc}
 \Gamma & & (\xi, x, y) \\
 \pi_1 \times \pi_2 \downarrow & \searrow \pi_3 & \downarrow \pi_1 \times \pi_2 \quad \searrow \pi_3 \\
 \mathbf{P}^1 \times G/P & \xrightarrow[\Phi]{} & G/P \\
 & & (\xi, x) \xrightarrow[\Phi]{} \Phi(\xi, x)
 \end{array}$$

The morphism  $\pi_1 \times \pi_2 : \Gamma \rightarrow \mathbf{P}^1 \times G/P$  is surjective, and restricts to an isomorphism between  $\mathcal{U}$  and  $\mathbf{C} \times G/P$ . In particular,  $\Gamma$  is an irreducible projective variety of dimension  $N + 1$ .

Likewise, let  $\mathcal{U}_w = \{(\xi, x, \Phi(\xi, x)) \mid \xi \in \mathbf{C}, x \in X_w\}$  and  $\Gamma_w$  be its closure, and similarly for  $\mathcal{U}^v, \mathcal{U}_w^v, \Gamma^v, \Gamma_w^v$ . Then  $\pi_3(\Gamma_w) = \pi_3(\overline{\mathcal{U}_w}) = \overline{\pi_3(\mathcal{U}_w)}$  in  $G/P$ , so  $\pi_3(\Gamma_w) \cap \mathcal{O}_\tau$  is the closure of  $\pi_3(\mathcal{U}_w) \cap \mathcal{O}_\tau = \Phi(\mathbf{C} \times Y_w)$  in  $\mathcal{O}_\tau$ . Proceeding similarly with  $\Gamma^v$  and  $\Gamma_w^v$ , we obtain

$$\pi_3(\Gamma_w) \cap \mathcal{O}_\tau = F_w, \quad \pi_3(\Gamma^v) \cap \mathcal{O}_\tau = F^v, \quad \pi_3(\Gamma_w^v) \cap \mathcal{O}_\tau = F_w^v.$$

We now need to study the  $\pi_3$ -fibre of a point in  $F_w$ . Actually, if  $y$  is in  $Y_w$ , then its fibre lies entirely in  $\Gamma_w$ . Indeed,  $U_\tau^-$  naturally acts on  $G/P$  and on  $\Gamma$  via  $g.(\xi, x, y) = (\xi, g.x, g.y)$  (since  $U_\tau^-$  is Abelian), and the morphism  $\pi_3$  is  $U_\tau^-$ -equivariant. It follows that whenever two points in  $G/P$  belong to the same  $U_\tau^-$ -orbit, their fibres are isomorphic. Now since  $\pi_3 : \Gamma \rightarrow G/P$  is dominant, there is an open set in  $G/P$  in which every point has a fibre of pure dimension 1. Since  $\mathcal{O}_\tau$  is open in  $G/P$ , it meets this open set, and since  $\mathcal{O}_\tau$  is a  $U_\tau^-$ -orbit in  $G/P$ ,  $y$  itself has a fibre of pure dimension 1.

Now fix an irreducible component  $C$  of  $\pi_3^{-1}(y)$ . Then

$$(\pi_1 \times \pi_2(C)) \cap (\mathbf{C} \times G/P) \subset \Phi^{-1}(y).$$

If  $C \cap \mathcal{U} \neq \emptyset$ , then the left hand side of this inclusion is non-empty and of dimension 1. Since  $\Phi^{-1}(y)$  is isomorphic to the  $\mathbf{C}$ -orbit of  $y$ , it is itself irreducible of dimension (at most) 1, hence the inclusion becomes an equality. Taking closures, we then obtain  $C = \overline{\{(\xi, x, y) \mid (\xi, x) \in \Phi^{-1}(y)\}}$ ; in particular,  $C$  is the unique irreducible component of  $\pi_3^{-1}(y)$  that intersects  $\mathcal{U}$ . Note also that  $C \subset \Gamma_w$ .

Now let  $C'$  be an irreducible component of  $\pi_3^{-1}(y)$  different from  $C$ , so that  $C' \subset \{\infty\} \times G/P \times \{y\}$ . Let  $\Gamma_\infty \subset \Gamma$  be the subvariety  $\pi_1^{-1}(\infty)$ . We have a  $U_\tau^-$ -equivariant morphism  $\pi : \Gamma_\infty \rightarrow G/P : (\infty, x, y) \mapsto y$ , so  $C'$  is an irreducible subvariety of the fibre  $\pi^{-1}(y)$ . Since  $\Gamma_\infty \subsetneq \Gamma$ , its dimension is at most  $N$ . Because of the equivariance of  $\pi$ , we see that  $\mathcal{O}_\tau$  is in the image of  $\pi$ , so  $\pi$  is surjective. Decomposing  $\Gamma_\infty$  into irreducible components  $\Gamma_\infty = C_1 \cup \dots \cup C_r$ , we obtain  $G/P = \pi(C_1) \cup \dots \cup \pi(C_r)$ , so that for some  $i$ ,  $\pi : C_i \rightarrow G/P$  is onto. Renumbering the  $C_i$ , we may assume that for some  $t \geq 1$ ,  $C_1, \dots, C_t$  are mapped surjectively to  $G/P$ , and  $C_{t+1}, \dots, C_r$  are not. For  $i \leq t$ , there is an open set  $U_i$  of  $G/P$  such that each element on  $U_i$  has a finite fibre in  $C_i$ . For  $i > t$ , let  $U_i$  be the open set  $G/P \setminus \pi(C_i)$ . Taking the intersection  $U = \bigcap_{i=1}^n U_i$ , we obtain a non-empty open set of  $G/P$  satisfying the following property: for each  $z \in U$ , the fibre of  $z$  in  $\Gamma_\infty$  consists of a finite number of points. Again,  $U$  meets the open orbit  $\mathcal{O}_\tau$ , so this property is true for every point in  $\mathcal{O}_\tau$ , in particular for  $y$ . So  $C'$  is included in the finite fibre  $\pi^{-1}(y)$ : a contradiction. Therefore,  $C'$  cannot exist, i.e.  $\pi_3^{-1}(y) = C \subset \Gamma_w$  is irreducible, and not contained in  $\{\infty\} \times G/P \times G/P$ .

Assume now that  $F_w^v \neq F_w \cap F^v$ . By Proposition 4.4,  $F_w^v$  and  $F_w \cap F^v$  have the same dimension, thus  $F_w \cap F^v$  is not irreducible. Let  $F$  be an irreducible component of the intersection  $F_w \cap F^v$  different from  $F_w^v$ . Let  $y \in F$ , and assume that  $y \notin F_w^v$ . Then  $y \notin \pi_3(\mathcal{U}^v)$ , so  $\pi_3^{-1}\{y\} \subset \Gamma^v \setminus \mathcal{U}^v \subset \{\infty\} \times G/P \times G/P$ . But  $y \in F_w$ , and we have seen that in this case  $\pi_3^{-1}(y)$  is never contained in  $\{\infty\} \times G/P \times G/P$ . This gives a contradiction.  $\square$

#### APPENDIX. SINGULARITIES OF SCHUBERT VARIETIES IN $SO(2n+1)/P_1$

In this Appendix, we shall determine the singular locus of Schubert varieties in  $G/P$ , where  $G$  is of type  $B_n$  and  $P$  is cominuscule. So let  $V = \mathbf{C}^{2n+1}$  together with a non-degenerate symmetric bilinear form  $(\cdot, \cdot)$  given in the canonical basis  $(e_1, \dots, e_{2n+1})$  by the anti-diagonal matrix  $E$  with 1's all along the anti-diagonal. The expression of the quadratic form  $Q$  associated with  $(\cdot, \cdot)$  is

$$Q(x_1, \dots, x_{2n+1}) = x_{n+1}^2 + 2 \sum_{i=1}^n x_i x_{2n+2-i}.$$

Let  $G = SO(V)$ ,  $B \subset G$  the subgroup of upper triangular matrices, and  $T \subset G$  the subgroup of diagonal matrices. Then  $B$  is a Borel subgroup of  $G$  and  $T$  is a maximal torus of  $G$ . The group  $G$  acts naturally on  $V$ , and  $e_1$  is a highest weight vector, of weight  $\omega_1$  (the unique cominuscule weight of  $G$ ), so that  $G/P_1$  gets identified with the  $G$ -orbit of the line generated by  $e_1$ :

$$G/P_1 = \{[x_1 : \dots : x_{2n+1}] \mid Q(x_1, \dots, x_{2n+1}) = 0\}$$

In this setting, the Schubert varieties are given by

$$X_i = \{[x_1 : \dots : x_i : 0 : \dots : 0] \mid Q(x_1, \dots, x_i, 0, \dots, 0) = 0\},$$

with  $1 \leq i \leq 2n+1$ , but  $i \neq n+1$ . Indeed, let  $x = [x_1 : \dots : x_{i-1} : 1 : 0 : \dots : 0]$  with  $Q(x) = 0$ , and let us prove that  $x \in C_i$ , that is, there exists  $b \in B$  such that  $x = b.e_i$ . A straightforward calculation shows that we may take the columns  $b_1, \dots, b_{2n+1}$  of  $b$  as follows:

- Case 1:  $i < n+1$ .

$$b_j = \begin{cases} e_j & \text{if } j \neq i \text{ and } j \leq 2n+2-i \\ x & \text{if } j = i \\ e_j - x_{2n+2-j}e_{2n+2-i} & \text{otherwise} \end{cases}$$

- Case 2:  $i > n+1$ .

$$b_j = \begin{cases} e_j & \text{if } j \leq 2n+2-i \\ x & \text{if } j = i \\ e_j - x_{2n+2-j}e_{2n+2-i} & \text{otherwise} \end{cases}$$

The Jacobian criterion easily shows that  $\text{Sing } X_i$  is equal to  $X_{2n+1-i}$  if  $i > n+1$ , and empty if  $i < n+1$ . Moreover, since  $X_i$  is defined by a single quadratic equation, the multiplicity of a singular point must be equal to 2. Hence there are two cases for the multiplicity  $\mu_i(x)$  of a point  $x = [x_1 : \dots : x_i : 0 : \dots : 0]$  on  $X_i$ :

- Case 1:  $i < n+1$ . Then  $\mu_i(x) = 1$ .

- Case 2:  $i > n + 1$ . Then

$$\mu_i(x) = \begin{cases} 2 & \text{if } x_i = \cdots = x_{2n+2-i} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Of course, we have the same result for the opposite Schubert varieties

$$X^j = \{[0 : \cdots : 0 : x_j : \cdots : x_{2n+1}] \mid Q(0, \dots, 0, x_j, \dots, x_{2n+1}) = 0\}$$

There are again two cases for the multiplicity  $\mu^j(x)$  of  $x = [0 : \cdots : 0 : x_j : \cdots : x_{2n+1}]$  on  $X^j$ :

- Case 1:  $j < n + 1$ . Then

$$\mu^j(x) = \begin{cases} 2 & \text{if } x_j = \cdots = x_{2n+2-j} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

- Case 2:  $j > n + 1$ . Then  $\mu^j(x) = 1$ .

Note that a Richardson variety  $X_i^j$  ( $j \leq i$ ) also is a quadric in a projective space, so the multiplicity of a point  $m \in X_i^j$  must be at most 2. But by Theorem 0.1, if  $m$  were singular in both  $X_i$  and in  $X^j$ , then its multiplicity would be 4. This means that  $\text{Sing } X_i \cap \text{Sing } X^j = \emptyset$ , a fact that can also be verified directly: indeed, if this intersection is non-empty, then  $2n+3-j \leq 2n+1-i$ , so  $j \leq i \leq j-2$ , a contradiction.

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